A cool title

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Summary

Conservation and dissipation

Kinetic energy $\mathcal{K}$, enstrophy $\mathcal{E}$, helicity $\mathcal{H}$:

$$\mathcal{K} = \frac{1}{2} \langle \boldsymbol{u}, \boldsymbol{u} \rangle_{\Omega}, \quad \mathcal{E} = \frac{1}{2} \langle \boldsymbol{\omega}, \boldsymbol{\omega} \rangle_{\Omega}, \quad \mathcal{H} = \langle \boldsymbol{u}, \boldsymbol{\omega} \rangle_{\Omega}.$$

Mass conservation: $\nabla \cdot \boldsymbol{u} = 0$

Given a conservative external body force $\boldsymbol{f}$,

Kinetic energy conservation and dissipation:

In inviscid case $\nu = 0$, $\dfrac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = 0$; in viscous case $\nu \neq 0$, $\dfrac{\mathrm{d}\mathcal{K}}{\mathrm{d}t} = -2\nu\mathcal{E}$.

Helicity conservation and dissipation (or generation):

In inviscid case $\nu = 0$, $\dfrac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0$; in viscous case $\nu \neq 0$, $\dfrac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = -2\nu \langle \boldsymbol{\omega}, \nabla \times \boldsymbol{\omega} \rangle_{\Omega}$.

Dual field: Hilbert spaces

$$
\begin{tikzcd}
\mathbb{R} \arrow[r, hook] & H^1(\Omega) \arrow[r, "\nabla"] \arrow[d, leftrightarrow, "\star"'] & H(\mathrm{curl};\Omega) \arrow[r, "\nabla\times"] \arrow[d, leftrightarrow, "\star"'] & H(\mathrm{div};\Omega) \arrow[r, "\nabla\cdot"] \arrow[d, leftrightarrow, "\star"'] & L^2(\Omega) \arrow[r] \arrow[d, leftrightarrow, "\star"'] & 0 \\
0 & L^2(\Omega) \arrow[l] & H(\mathrm{div};\Omega) \arrow[l, "\nabla\cdot"'] & H(\mathrm{curl};\Omega) \arrow[l, "\nabla\times"'] & H^1(\Omega) \arrow[l, "\nabla"'] & \mathbb{R} \arrow[l, hook']
\end{tikzcd}
$$

Figure: A double de Rham complex of Hilbert spaces.

MEEVC [1]

"A *mass, energy, enstrophy* and *vorticity* conserving (MEEVC) mimetic spectral element discretization for the 2D incompressible Navier-Stokes equations" [1]

A system of two evolution equations are used in MEEVC scheme:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \omega \times \boldsymbol{u} + \nu \nabla \times \omega + \nabla P = 0, \tag{1a}$$

$$\frac{\partial \omega}{\partial t} + \frac{1}{2}(\boldsymbol{u} \cdot \nabla)\omega + \frac{1}{2}\nabla \cdot (\boldsymbol{u}\omega) = \Delta\omega, \tag{1b}$$

$$\nabla \cdot \boldsymbol{u} = 0. \tag{1c}$$

Staggered temporal discretization

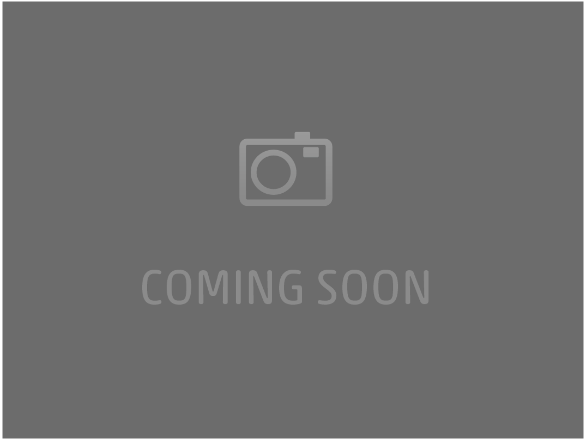

Figure: caption

Dissipation properties after temporal discretization

Given a conservative external body force, when $\nu \neq 0$, by repeating aforementioned analysis, one now can find that kinetic energy dissipates at the rate

$$\frac{\mathcal{K}_1^{k+\frac{1}{2}} - \mathcal{K}_1^{k-\frac{1}{2}}}{\Delta t} = -\nu \left\langle \frac{\boldsymbol{\omega}_2^{k-\frac{1}{2}} + \boldsymbol{\omega}_2^{k+\frac{1}{2}}}{2}, \frac{\boldsymbol{\omega}_2^{k-\frac{1}{2}} + \boldsymbol{\omega}_2^{k+\frac{1}{2}}}{2} \right\rangle_\Omega = -2\nu\mathcal{E}_2^k \leq 0,$$

$$\frac{\mathcal{K}_2^{k} - \mathcal{K}_2^{k-1}}{\Delta t} = -\nu \left\langle \frac{\boldsymbol{\omega}_1^{k-1} + \boldsymbol{\omega}_1^{k}}{2}, \frac{\boldsymbol{\omega}_1^{k-1} + \boldsymbol{\omega}_1^{k}}{2} \right\rangle_\Omega = -2\nu\mathcal{E}_1^{k+\frac{1}{2}} \leq 0.$$

And helicity dissipates or generates at the rate:

$$\frac{\mathcal{H}_1^{k} - \mathcal{H}_1^{k-1}}{\Delta t} = \frac{\mathcal{H}_2^{k} - \mathcal{H}_2^{k-1}}{\Delta t} = -\nu \left\langle \nabla \times \omega_1^{k-\frac{1}{2}}, \omega_2^{k-\frac{1}{2}} \right\rangle_\Omega - \nu \frac{\left\langle \omega_2^k, \nabla \times \omega_1^k \right\rangle_\Omega + \left\langle \omega_2^{k-1}, \nabla \times \omega_1^{k-1} \right\rangle_\Omega}{2}.$$

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$$\frac{\mathcal{K}_1^{k+\frac{1}{2}} - \mathcal{K}_1^{k-\frac{1}{2}}}{\Delta t} = -\nu \left\langle \frac{\boldsymbol{\omega}_2^{k-\frac{1}{2}} + \boldsymbol{\omega}_2^{k+\frac{1}{2}}}{2}, \frac{\boldsymbol{\omega}_2^{k-\frac{1}{2}} + \boldsymbol{\omega}_2^{k+\frac{1}{2}}}{2} \right\rangle_\Omega = -2\nu \mathcal{E}_2^k \leq 0,$$

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Mimetic spatial discretization

In order to ensure the conservation at the fully discrete level, finite dimensional function spaces employed for the spatial discretization need to form a **discrete de Rham complex** [2],

$$\mathbb{R} \hookrightarrow H^1(\Omega^h) \xrightarrow{\nabla} H(\mathrm{curl};\Omega^h) \xrightarrow{\nabla\times} H(\mathrm{div};\Omega^h) \xrightarrow{\nabla\cdot} L^2(\Omega^h) \rightarrow 0,$$

We called these spaces *structure-preserving* or *mimetic* spaces.

✓ Once this is the case, the proofs for the conservation properties and the derivations for kinetic energy dissipation and helicity dissipation (or generation) at the semi-discrete level must hold at the fully discrete level.
□

We have used the *mimetic polynomial spaces* as our mimetic spaces for the tests.

Test 2: Manufactured convergence test

In periodic unit cube $\Omega := [0,1]^3$, we use

$$\boldsymbol{u} = \{(2-t)\cos(2\pi z),\ (1+t)\sin(2\pi z),\ (1-t)\sin(2\pi x)\}^{\mathsf{T}}$$

and

$$p = \sin(2\pi(x+y+t))$$

as exact solutions ($\boldsymbol{\omega}$ and $\boldsymbol{f}$ then can be calculated) and compute the flow from $t=0$ to $t=2$ and measure the error at $t=2$.

References

- A. Palha, M. Gerritsma, A mass, energy, enstrophy and vorticity conserving (MEEVC) mimetic spectral element discretization for the 2D incompressible Navier–Stokes equations, Journal of Computational Physics 328 (2017) 200–220.
- Y. Zhang, A. Palha, M. Gerritsma, L. G. Rebholz, A mass-, kinetic energy- and helicity-conserving mimetic dual-field discretization for three-dimensional incompressible Navier-Stokes equations, part I: Periodic domains, Journal of Computational Physics 451 (2022) 110868.