A hybrid mimetic spectral element method for the vorticity-velocity-pressure formulation of the Stokes equations

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Mimetic

Mimetic spectral element method (MSEM) [1, 2, 3] is arbitrary order structure-preserving mixed finite element methods.

It is structure-preserving because they use finite dimensional function spaces that obey the **de Rham complex** :

$$\begin{array}{ccccccccc} \mathbb{R} \to & H^1 & \overset{\text{grad}}{\to} & H(\text{curl}) & \overset{\text{curl}}{\to} & H(\text{div}) & \overset{\text{div}}{\to} & L^2 \to 0, \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{R} \to & H_h^1 & \overset{\text{grad}}{\to} & H_h(\text{curl}) & \overset{\text{curl}}{\to} & H_h(\text{div}) & \overset{\text{div}}{\to} & L_h^2 \to 0. \end{array}$$

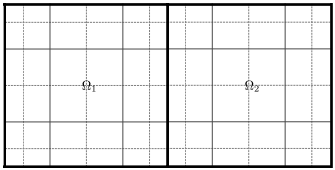
It is computationally costly. (i) : the large amount of dofs, (ii) : the low sparsity.

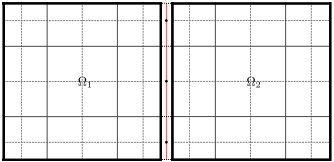
We are going to reduce the computation cost of MSEM from this two aspects.

1. Kreeft, J., Palha, A. and Gerritsma, M. Mimetic framework on curvilinear quadrilaterals of arbitrary order. *arXiv preprint*, (2011) arXiv :1111.4304.
2. Kreeft, J. and Gerritsma, M. Mixed mimetic spectral element method for Stokes flow : A pointwise divergence-free solution. *Journal of Computational Physics*, (2013) 240 : 284-309.
3. Palha, A., Rebelo, P.P., Hiemstra, R., Kreeft, J. and Gerritsma, M. Physics-compatible discretization techniques on single and dual grids, with application to the Poisson equation of volume forms. *Journal of Computational Physics*, (2014) 257 : 1394-1422.

Hybrid finite element methods

Hybrid finite element methods [4 5] are those methods that first allow the discontinuity across the inter-element interface then re-enforce (weakly or strongly) the continuity by introducing a Lagrange multiplier between elements.





↑
Lagrange multiplier

Similar idea has also be used in, e.g., *mortar methods* and *finite element tearing and interconnecting (FETI) methods*, and more.

4. T. H. Pian, C.-C. Wu, Hybrid and incompatible finite element methods, Chapman and Hall/CRC, 2005.
5. F. Brezzi, M. Fortin, Mixed and hybrid finite element methods, Vol. 15, Springer Science & Business Media, 2012.

Dual basis functions

Using dual basis functions[6] [7] eliminates some metric-dependent matrices from the discrete system.

$$(\phi, \varphi) = \underline{\phi}^{\mathsf{T}} \mathbb{M} \underline{\varphi}$$

$$(\phi, \widetilde{\varphi}) = \underline{\phi}^{\mathsf{T}} \underline{\widetilde{\varphi}}$$

where $\widetilde{\varphi}$ represent it is expanded with dual basis functions.

6. P. Wozny, Construction of dual bases, Journal of Computational and Applied Mathematics 245 (2013) 75-85
7. P. Wozny, Construction of dual b-spline functions, Journal of Computational and Applied Mathematics 260 (2014) 301-311

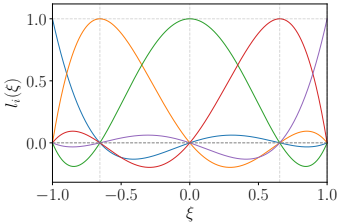
Mimetic basis functions

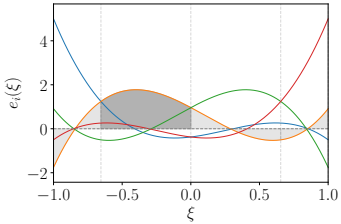
Let $-1 \leq \xi_0 < \xi_1 < \cdots < \xi_N \leq 1$. The well-known Lagrange polynomials are expressed by $l_i(\xi)$:

$$l_i(\xi) = \prod_{j=0, j\neq i}^{N} \frac{\xi - \xi_j}{\xi_i - \xi_j}, \quad i \in \{0, 1, 2, \cdots, N\}, \text{ satisfying, } l_i(\xi) = \delta_{i,j}.$$

The corresponding **edge polynomials**[8] are

$$e_i(\xi) = -\sum_{k=0}^{i-1} \frac{\mathrm{d}l_k(\xi)}{\mathrm{d}\xi} = \sum_{k=i}^{N} \frac{\mathrm{d}l_k(\xi)}{\mathrm{d}\xi}, \; i \in \{1, 2, \cdots, N\} \text{ satisfying, } \int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \delta_{i,j}.$$





8. Gerritsma, M. Edge functions for spectral element methods. *Spectral and High Order Methods for Partial Differential Equations.* Springer, (2011) 199-207

Mimetic basis functions

Let $\mathtt{L}^N := \mathrm{span}(l_i)$, and $\mathtt{E}^{(N-1)} := \mathrm{span}(e_i)$. If $p_h \in \mathtt{L}^N$, $q_h \in \mathtt{E}^{(N-1)}$, and $q_h = \mathrm{d}p_h$

$$p_h = \sum_{i=0}^{N} \mathsf{p}_i l_i(\xi) \quad \text{and} \quad q_h = \sum_{i=1}^{N} \mathsf{q}_i e_i(\xi)$$

Let $\underline{p} = \begin{Bmatrix} \mathsf{p}_0 & \mathsf{p}_1 & \cdots & \mathsf{p}_N \end{Bmatrix}^\mathsf{T}$, $\underline{q} = \begin{Bmatrix} \mathsf{q}_1 & \mathsf{q}_2 & \cdots & \mathsf{q}_N \end{Bmatrix}^\mathsf{T}$. We have $\underline{q} = \mathbb{E}\underline{p}$,

$$\mathbb{E} = \begin{Bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{Bmatrix}$$

We have constructed such a discrete de Rham complex in $\mathbb{R}$:

$$\begin{array}{ccccc} \underline{\mathtt{L}}^N & \leftrightarrow & \mathtt{L}^N & \subset & H^1(I) \\ \downarrow \mathbb{E} & & \downarrow \mathrm{d} & & \downarrow \mathrm{d}. \\ \underline{\mathtt{E}}^{(N-1)} & \leftrightarrow & \mathtt{E}^{(N-1)} & \subset & L^2(I) \end{array}$$

Mimetic basis functions

In $\mathbb{R}^3$, we define spaces

$$
\begin{aligned}
\mathscr{N} &:= \mathtt{L}^N \otimes \mathtt{L}^N \otimes \mathtt{L}^N, \\
\mathscr{E} &:= \mathtt{E}^{(N-1)} \otimes \mathtt{L}^N \otimes \mathtt{L}^N \quad \times \quad \mathtt{L}^N \otimes \mathtt{E}^{(N-1)} \otimes \mathtt{L}^N \quad \times \quad \mathtt{L}^N \otimes \mathtt{L}^N \otimes \mathtt{E}^{(N-1)}, \\
\mathscr{S} &:= \mathtt{L}^N \otimes \mathtt{E}^{(N-1)} \otimes \mathtt{E}^{(N-1)} \quad \times \quad \mathtt{E}^{(N-1)} \otimes \mathtt{L}^N \otimes \mathtt{E}^{(N-1)} \quad \times \quad \mathtt{E}^{(N-1)} \otimes \mathtt{E}^{(N-1)} \otimes \mathtt{L}^N, \\
\mathscr{V} &:= \mathtt{E}^{(N-1)} \otimes \mathtt{E}^{(N-1)} \otimes \mathtt{E}^{(N-1)},
\end{aligned}
$$

where notations $\mathscr{N}$, $\mathscr{E}$, $\mathscr{S}$, and $\mathscr{V}$ stand for *nodes*, *edges*, *surfaces*, and *volumes* because the degrees of freedom are associated with nodes, edges, faces and volumes, due to the Kronecker delta properties of the nodal and edge functions.

$$
\begin{array}{ccccc}
\underline{\mathscr{N}} & \leftrightarrow & \mathscr{N} & \subset & H^1(\Omega_{\text{ref}}) \\
\downarrow \mathbb{E}_{\text{grad}} & & \downarrow \text{grad} & & \downarrow \text{grad} \\
\underline{\mathscr{E}} & \leftrightarrow & \mathscr{E} & \subset & H(\text{curl};\Omega_{\text{ref}}) \\
\downarrow \mathbb{E}_{\text{curl}} & & \downarrow \text{curl} & & \downarrow \text{curl} \\
\underline{\mathscr{S}} & \leftrightarrow & \mathscr{S} & \subset & H(\text{div};\Omega_{\text{ref}}) \\
\downarrow \mathbb{E}_{\text{div}} & & \downarrow \text{div} & & \downarrow \text{div} \\
\underline{\mathscr{V}} & \leftrightarrow & \mathscr{V} & \subset & L^2(\Omega_{\text{ref}})
\end{array}
.
$$

Mimetic basis functions

We consider the discrete vector valued function $\boldsymbol{\alpha}^h$ in $\mathscr{S}$. The trace of $\boldsymbol{\alpha}^h$ on the face, for example, $(\xi, \eta, \varsigma) \in \Gamma_{\xi^-} = -1 \times [-1, 1] \times [-1, 1]$ is

$$\mathrm{tr}_{\xi^-} \boldsymbol{\alpha}^h = \begin{Bmatrix} \alpha^h_\xi(-1, \eta, \varsigma) \\ \alpha^h_\eta(-1, \eta, \varsigma) \\ \alpha^h_\varsigma(-1, \eta, \varsigma) \end{Bmatrix} \cdot \begin{Bmatrix} -1 \\ 0 \\ 0 \end{Bmatrix} = -\sum_{i=0}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \mathsf{a}^{\xi}_{i,j,k} l_i(-1) e_j(\eta) e_k(\varsigma) = \sum_{j=1}^{N} \sum_{k=1}^{N} \mathsf{a}^{\xi^-}_{j,k} e_j(\eta) e_k(\varsigma).$$

The polynomials $e_j(\eta) e_k(\varsigma)$ then span a trace space on Γ_{ξ^-}. We denote this trace space by $\mathcal{S}_{\xi^-}$ and there is a linear operator $\mathbb{N}_{\xi^-}$ which maps $\underline{\boldsymbol{\alpha}}$ into $\underline{\alpha}_{\xi^-}$:

$$\underline{\alpha}_{\xi^-} = \mathbb{N}_{\xi^-} \underline{\boldsymbol{\alpha}}.$$

For example, if $N = 2$, we can have

$$\mathbb{N}_{\xi^-} = \begin{Bmatrix} -1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & 0 & \cdots \end{Bmatrix}$$

Dual representations

We consider the primal polynomial space $\mathscr{V}$, and let $\varphi^h, \phi^h \in \mathscr{V}$. The L^2-inner product between φ^h and ϕ^h is

$$\left(\varphi^h, \phi^h\right)_{\Omega_{\text{ref}}} = \underline{\varphi}^{\mathsf{T}} \mathbb{M}_{\mathscr{V}} \underline{\phi},$$

The dual polynomials can then be defined as

$$\left\{\cdots, \widetilde{ee}e_{i,j,k}(\xi,\eta,\varsigma), \cdots\right\}^{\mathsf{T}} := \mathbb{M}_{\mathscr{V}}^{-1} \left\{\cdots, e_i(\xi)e_j(\eta)e_k(\varsigma), \cdots\right\}^{\mathsf{T}}.$$

These dual polynomials form another basis of the space $\mathscr{V}$. An element, ϕ^h, in $\mathscr{V}$ has a unique dual representation, denoted by $\widetilde{\phi}^h$, whose degrees of freedom are

$$\underline{\widetilde{\phi}} = \mathbb{M}_{\mathscr{V}} \underline{\phi}.$$

Now, the L^2-inner product between φ^h and $\widetilde{\phi}^h$ is

$$\left(\varphi^h, \widetilde{\phi}^h\right)_{\Omega_{\text{ref}}} = \underline{\varphi}^{\mathsf{T}} \underline{\widetilde{\phi}}.$$

Dual representations

1. If $\beta^h = \mathrm{div}\boldsymbol{\alpha}^h \in \mathscr{V}$ and $\widetilde{\phi}^h \in \mathscr{V}$, we have

$$\left(\widetilde{\phi}^h, \beta^h\right)_{\Omega_{\mathrm{ref}}} = \left(\widetilde{\phi}^h, \mathrm{div}\boldsymbol{\alpha}^h\right)_{\Omega_{\mathrm{ref}}} = \underline{\widetilde{\phi}}^{\mathsf{T}} \mathbb{E}_{\mathrm{div}} \underline{\boldsymbol{\alpha}}. \tag{1}$$

2. If $\boldsymbol{\alpha}^h \in \mathscr{S}$ and $\widetilde{\gamma} \in \mathcal{S}_{\xi^-}$, we have

$$\left\langle \widetilde{\gamma}, \mathrm{tr}_{\xi^-} \boldsymbol{\alpha}^h \right\rangle_{\Gamma_{\xi^-}} = \underline{\widetilde{\gamma}}^{\mathsf{T}} \mathbb{N}_{\xi^-} \underline{\boldsymbol{\alpha}}. \tag{2}$$

3. A integration by parts is given by

$$\int_\Omega \boldsymbol{\alpha} \cdot \mathrm{grad}\phi \mathrm{d}\Omega = \int_{\partial\Omega} \widehat{\phi}\boldsymbol{\alpha} \cdot \boldsymbol{n} \mathrm{d}\Gamma - \int_\Omega \phi \ \mathrm{div}\boldsymbol{\alpha} \mathrm{d}\Omega.$$

If $\boldsymbol{\alpha}^h \in \mathscr{S}$ and $\widetilde{\phi}^h \in \mathscr{V}$, the discrete integration by parts is written as

$$\left(\mathrm{grad}\widetilde{\phi}^h, \boldsymbol{\alpha}^h\right)_{\Omega_{\mathrm{ref}}} = \left\langle \widetilde{\phi}^h, \mathrm{tr}\boldsymbol{\alpha}^h \right\rangle_{\partial\Omega_{\mathrm{ref}}} - \left(\widetilde{\phi}^h, \mathrm{div}\boldsymbol{\alpha}^h\right)_{\Omega_{\mathrm{ref}}}. \tag{3}$$

If $\widetilde{\vartheta}^h := \nabla\widetilde{\phi}^h$, with (1) and (2), we know

$$\underline{\widetilde{\vartheta}} = \mathbb{N}_{\mathcal{S}}^{\mathsf{T}} \underline{\widehat{\phi}} - \mathbb{E}_{\mathrm{div}}^{\mathsf{T}} \underline{\widetilde{\phi}}, \tag{4}$$

where $\mathbb{N}_{\mathcal{S}}$ can be obtained by assembling the trace matrices.

Transformation

Let a $\mathcal{C}^1$ diffeomorphism, $\Phi_m : \Omega_{\text{ref}} \to \Omega_m$, and use $\mathscr{N}_m$, $\mathscr{E}_m$, $\mathscr{S}_m$, and $\mathscr{V}_m$ represent the corresponding spaces in Ω_m.

1. The transformation between $\psi^h(\xi,\eta,\varsigma) \in \mathscr{N}$ and $\psi^h_m(x,y,z) \in \mathscr{N}_m$ is given by

$$\psi^h_m(x,y,z) = \left(\psi^h \circ \Phi_m^{-1}\right)(x,y,z), \quad \psi^h(\xi,\eta,\varsigma) = \left(\psi^h_m \circ \Phi_m\right)(\xi,\eta,\varsigma).$$

2. The transformation between $\boldsymbol{\varphi}^h(\xi,\eta,\varsigma) \in \mathscr{E}$ and $\boldsymbol{\varphi}^h_m(x,y,z) \in \mathscr{E}_m$ is given by

$$\boldsymbol{\varphi}^h_m(x,y,z) = \left(\mathcal{J}^\mathsf{T}\right)^{-1}\left(\boldsymbol{\varphi}^h \circ \Phi_m^{-1}\right)(x,y,z), \quad \boldsymbol{\varphi}^h(\xi,\eta,\varsigma) = \mathcal{J}^\mathsf{T}\left(\boldsymbol{\varphi}^h_m \circ \Phi_m\right)(\xi,\eta,\varsigma).$$

3. The transformation between $\boldsymbol{\alpha}^h(\xi,\eta,\varsigma) \in \mathscr{S}$ and $\boldsymbol{\alpha}^h_m(x,y,z) \in \mathscr{S}_m$ is given by

$$\boldsymbol{\alpha}^h_m(x,y,z) = \frac{\mathcal{J}}{\det\mathcal{J}}\left(\boldsymbol{\alpha}^h \circ \Phi_m^{-1}\right)(x,y,z), \quad \boldsymbol{\alpha}^h(\xi,\eta,\varsigma) = \frac{\mathcal{J}^{-1}}{\det\mathcal{J}}\left(\boldsymbol{\alpha}^h_m \circ \Phi_m\right)(\xi,\eta,\varsigma).$$

4. The transformation between $\beta^h(\xi,\eta,\varsigma) \in \mathscr{V}$ and $\beta^h_m(x,y,z) \in \mathscr{V}_m$ is given by

$$\beta^h_m(x,y,z) = \frac{1}{\det\mathcal{J}}\left(\beta^h \circ \Phi_m^{-1}\right)(x,y,z), \quad \beta^h(\xi,\eta,\varsigma) = \det\mathcal{J}\left(\beta^h_m \circ \Phi_m\right)(\xi,\eta,\varsigma).$$

Transformation

Let a $\mathcal{C}^1$ diffeomorphism, $\Phi_m : \Omega_{\text{ref}} \to \Omega_m$, and use $\mathscr{N}_m$, $\mathscr{E}_m$, $\mathscr{S}_m$, and $\mathscr{V}_m$ represent the corresponding spaces in Ω_m.

$$
\begin{array}{ccccc}
\underline{\mathscr{N}}_m & \leftrightarrow & \mathscr{N}_m & \subset & H^1(\Omega_m) \\
\downarrow \mathbb{E}_{\text{grad}} & & \downarrow \text{grad} & & \downarrow \text{grad} \\
\underline{\mathscr{E}}_m & \leftrightarrow & \mathscr{E}_m & \subset & H(\text{curl};\Omega_m) \\
\downarrow \mathbb{E}_{\text{curl}} & & \downarrow \text{curl} & & \downarrow \text{curl} \\
\underline{\mathscr{S}}_m & \leftrightarrow & \mathscr{S}_m & \subset & H(\text{div};\Omega_m) \\
\downarrow \mathbb{E}_{\text{div}} & & \downarrow \text{div} & & \downarrow \text{div} \\
\underline{\mathscr{V}}_m & \leftrightarrow & \mathscr{V}_m & \subset & L^2(\Omega_m)
\end{array}
\quad .
$$

Stokes Velocity-Vorticity-Pressure formulation

The Stokes equations in the Velocity-Vorticity-Pressure formulation is given as

$$\begin{cases} \boldsymbol{\omega} - \nabla \times \boldsymbol{u} = 0 \\ \nabla \times \boldsymbol{\omega} + \nabla p = \boldsymbol{f} \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases}$$

Given a bounded domain Ω in $\mathbb{R}^3$ with boundaries $\partial\Omega = \Gamma_p \cup \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}} = \Gamma_{\boldsymbol{u}\times\boldsymbol{n}} \cup \Gamma_{\boldsymbol{\omega}}$ where $\Gamma_p \cap \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}} = \emptyset$, $\Gamma_{\boldsymbol{u}\times\boldsymbol{n}} \cap \Gamma_{\boldsymbol{\omega}} = \emptyset$, and given $\widehat{p}$ on Γ_p, $\widehat{\boldsymbol{u}} \cdot \boldsymbol{n}$ on $\Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}}$, $\widehat{\boldsymbol{u}} \times \boldsymbol{n}$ on $\Gamma_{\boldsymbol{u}\times\boldsymbol{n}}$, and $\widehat{\boldsymbol{\omega}}$ on $\Gamma_{\boldsymbol{\omega}}$.

$$\mathcal{L}(\boldsymbol{\omega}, \boldsymbol{u}, p; \boldsymbol{f}, g, \widehat{\boldsymbol{u}} \times \boldsymbol{n}, \widehat{p}) = \frac{1}{2}(\boldsymbol{\omega}, \boldsymbol{\omega})_\Omega - \langle \boldsymbol{\omega}, \widehat{\boldsymbol{u}} \times \boldsymbol{n} \rangle_{\Gamma_{\boldsymbol{u}\times\boldsymbol{n}}} - (\boldsymbol{u}, \nabla \times \boldsymbol{\omega} - \boldsymbol{f})_\Omega + (p, \nabla \cdot \boldsymbol{u})_\Omega - \langle \widehat{p}, \boldsymbol{u} \cdot \boldsymbol{n} \rangle_{\Gamma_p}, \tag{5}$$

where $\boldsymbol{\omega} \in H_0(\mathrm{curl}, \Omega; \Gamma_{\boldsymbol{\omega}})$, $\boldsymbol{u} \in H_0(\mathrm{div}, \Omega; \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}})$, $p \in L^2(\Omega)$, and $f \in [L^2(\Omega)]^3$, $g \in L^2(\Omega)$, $\widehat{\boldsymbol{u}} \times \boldsymbol{n} \in TH_{00}^{1/2}(\Gamma_{\boldsymbol{u}\times\boldsymbol{n}})$, $\widehat{p} \in H_{00}^{1/2}(\Gamma_p)$ are given.

$$\mathrm{tr}\ \boldsymbol{\omega} \in \left[TH_{00}^{1/2}(\Gamma_{\boldsymbol{u}\times\boldsymbol{n}})\right]'$$

$$\mathrm{tr}\ \boldsymbol{u} \in \left[H_{00}^{1/2}(\Gamma_p)\right]'$$

Variational formulation

Given $f \in [L^2(\Omega)]^3, g \in L^2(\Omega), \widehat{\boldsymbol{u}} \times \boldsymbol{n} \in TH^{1/2}_{00}(\Gamma_{\boldsymbol{u}\times\boldsymbol{n}}), \widehat{p} \in H^{1/2}_{00}(\Gamma_p)$, seek $\boldsymbol{\omega} \in H_0(\mathrm{curl}, \Omega; \Gamma_{\boldsymbol{\omega}}), \boldsymbol{u} \in H_0(\mathrm{div}, \Omega; \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}})$, $p \in L^2(\Omega)$ such that

$$\begin{cases} (\boldsymbol{\omega}, \overline{\boldsymbol{\omega}})_\Omega - (\boldsymbol{u}, \nabla \times \overline{\boldsymbol{\omega}})_\Omega = \langle \mathrm{tr}\, \overline{\boldsymbol{\omega}}, \widehat{\boldsymbol{u}} \times \boldsymbol{n} \rangle_{\Gamma_{\boldsymbol{u}\times\boldsymbol{n}}}, & \forall \overline{\boldsymbol{\omega}} \in H_0(\mathrm{curl}, \Omega; \Gamma_{\boldsymbol{\omega}}) \\ (\overline{\boldsymbol{u}}, \nabla \times \boldsymbol{\omega})_\Omega - (p, \nabla \cdot \overline{\boldsymbol{u}})_\Omega = (\overline{\boldsymbol{u}}, \boldsymbol{f})_\Omega - \langle \widehat{p}, \overline{\boldsymbol{u}} \cdot \boldsymbol{n} \rangle_{\Gamma_p}, & \forall \overline{\boldsymbol{u}} \in H_0(\mathrm{div}, \Omega; \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}}) \\ (\overline{p}, \nabla \cdot \boldsymbol{u})_\Omega = 0, & \forall \overline{p} \in L^2(\Omega) \end{cases}.$$

We choose $\mathscr{E}$ to approximate $H_0(\mathrm{curl}, \Omega; \Gamma_{\boldsymbol{\omega}})$, $\mathscr{S}$ to approximate $H_0(\mathrm{div}, \Omega; \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}})$, $\mathscr{V}$ to approximate $L^2(\Omega)$, we obtain

$$\begin{Bmatrix} \mathbb{M}_{\mathscr{E}} & -\mathbb{E}^{\mathsf{T}}_{\mathrm{curl}}\mathbb{M}_{\mathscr{S}} & \mathbf{0} \\ -\mathbb{M}_{\mathscr{S}}\mathbb{E}_{\mathrm{curl}} & \mathbf{0} & \mathbb{E}^{\mathsf{T}}_{\mathrm{div}}\mathbb{M}_{\mathscr{V}} \\ \mathbb{M}_{\mathscr{V}}\mathbb{E}_{\mathrm{div}} & \mathbf{0} & \mathbf{0} \end{Bmatrix} \begin{Bmatrix} \boldsymbol{\omega} \\ \boldsymbol{u} \\ \boldsymbol{p} \end{Bmatrix} = \begin{Bmatrix} \mathbb{B}_{\widehat{\boldsymbol{u}}\times\boldsymbol{n}} \\ -\mathbb{M}_{\mathscr{S}}\boldsymbol{f} + \mathbb{B}_{\widehat{p}} \\ \mathbf{0} \end{Bmatrix}$$

Hybridization

Now, we consider to divide the domain Ω into a mesh of M disjoint subdomains, Ω_m, $m = 1, 2, \cdots, M$, and we use Γ_{mn} to represent the interface of Ω_m and Ω_n.

Given $f \in [L^2(\Omega_m)]^3$, $g \in L^2(\Omega_m)$, $\widehat{\boldsymbol{u}} \times \boldsymbol{n} \in TH_{00}^{1/2}(\Gamma_{\boldsymbol{u}\times\boldsymbol{n}} \cap \partial\Omega_m)$, $\widehat{p} \in H_{00}^{1/2}(\Gamma_p \cap \partial\Omega_m)$, seek $\boldsymbol{\omega} \in H_0(\mathrm{curl}, \Omega_m; \Gamma_{\boldsymbol{\omega}} \cap \partial\Omega_m)$, $\boldsymbol{u} \in H_0(\mathrm{div}, \Omega_m; \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}} \cap \partial\Omega_m)$, $p \in L^2(\Omega_m)$, $\gamma \in TH_{00}^{1/2}(\Gamma_{mn})$, $\lambda \in H_{00}^{1/2}(\Gamma_{mn})$, such that

$$
\begin{cases}
(\boldsymbol{\omega}, \overline{\boldsymbol{\omega}})_{\Omega_m} - (\boldsymbol{u}, \nabla \times \overline{\boldsymbol{\omega}})_{\Omega_m} - \sum_n \langle \mathrm{tr}\, \overline{\boldsymbol{\omega}}, \gamma \rangle_{\Gamma_{mn}} = \langle \mathrm{tr}\, \overline{\boldsymbol{\omega}}, \widehat{\boldsymbol{u}} \times \boldsymbol{n} \rangle_{\Gamma_{\boldsymbol{u}\times\boldsymbol{n}} \cap \partial\Omega_m}, & \forall \overline{\boldsymbol{\omega}} \in H_0(\mathrm{curl}, \Omega_m; \Gamma_{\boldsymbol{\omega}} \cap \partial\Omega_m) \\
(\overline{\boldsymbol{u}}, \nabla \times \boldsymbol{\omega})_{\Omega_m} - (p, \nabla \cdot \overline{\boldsymbol{u}})_{\Omega_m} + \sum_n \langle \lambda, \overline{\boldsymbol{u}} \cdot \boldsymbol{n} \rangle_{\Gamma_{mn}} = (\overline{\boldsymbol{u}}, \boldsymbol{f})_{\Omega_m} - \langle \widehat{p}, \overline{\boldsymbol{u}} \cdot \boldsymbol{n} \rangle_{\Gamma_p \cap \partial\Omega_m}, & \forall \overline{\boldsymbol{u}} \in H_0(\mathrm{div}, \Omega_m; \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}} \cap \partial\Omega_m) \\
(\overline{p}, \nabla \cdot \boldsymbol{u})_{\Omega} = 0, & \forall \overline{p} \in L^2(\Omega_m) \\
\langle \overline{\gamma}, \mathrm{tr}\, \boldsymbol{\omega}_m - \mathrm{tr}\, \boldsymbol{\omega}_n \rangle_{\Gamma_{mn}} = 0, & \forall \overline{\gamma} \in TH_{00}^{1/2}(\Gamma_{mn}) \\
\langle \overline{\lambda}, \boldsymbol{u}_m \cdot \boldsymbol{n} - \boldsymbol{u}_n \cdot \boldsymbol{n} \rangle_{\Gamma_{mn}} = 0, & \forall \overline{\lambda} \in H_{00}^{1/2}(\Gamma_{mn})
\end{cases}
.
$$

Hybridization

In the mixed formulation, we choose $\mathscr{E}$ to approximate $H_0(\mathrm{curl}, \Omega; \Gamma_{\boldsymbol{\omega}})$, $\mathscr{S}$ to approximate $H_0(\mathrm{div}, \Omega; \Gamma_{\boldsymbol{u}\cdot\boldsymbol{n}})$, $\mathscr{V}$ to approximate $L^2(\Omega)$, we obtain

$$\begin{Bmatrix} \mathbb{M}_{\mathscr{E}} & -\mathbb{E}_{\mathrm{curl}}^{\mathsf{T}}\mathbb{M}_{\mathscr{S}} & \mathbf{0} \\ -\mathbb{M}_{\mathscr{S}}\mathbb{E}_{\mathrm{curl}} & \mathbf{0} & \mathbb{E}_{\mathrm{div}}^{\mathsf{T}}\mathbb{M}_{\mathscr{V}} \\ \mathbb{M}_{\mathscr{V}}\mathbb{E}_{\mathrm{div}} & \mathbf{0} & \mathbf{0} \end{Bmatrix} \begin{Bmatrix} \vec{\boldsymbol{\omega}} \\ \vec{\boldsymbol{u}} \\ \vec{\boldsymbol{p}} \end{Bmatrix} = \begin{Bmatrix} \mathbb{B}_{\widehat{\boldsymbol{u}}\times\boldsymbol{n}} \\ -\mathbb{M}_{\mathscr{S}}\vec{\boldsymbol{f}} + \mathbb{B}_{\widehat{p}} \\ \mathbf{0} \end{Bmatrix}$$

In the hybrid mixed formulation, we further choose $\mathrm{tr}\mathscr{E}$ to approximate $TH_{00}^{1/2}(\Gamma_{mn})$, and $\mathrm{tr}\mathscr{S}$ to approximate $H_{00}^{1/2}(\Gamma_{mn})$

$$\begin{Bmatrix} \mathbb{M}_{\mathscr{E}} & -\mathbb{E}_{\mathrm{curl}}^{\mathsf{T}}\mathbb{M}_{\mathscr{S}} & \mathbf{0} & \mathbb{N}_{\mathscr{E}}^{\mathsf{T}}\mathbf{M}_{\mathrm{tr}\mathscr{E}} & \mathbf{0} \\ -\mathbb{M}_{\mathscr{S}}\mathbb{E}_{\mathrm{curl}} & \mathbf{0} & \mathbb{E}_{\mathrm{div}}^{\mathsf{T}}\mathbb{M}_{\mathscr{V}} & \mathbf{0} & \mathbb{N}_{\mathscr{S}}^{\mathsf{T}}\mathbf{M}_{\mathrm{tr}\mathscr{S}} \\ \mathbb{M}_{\mathscr{V}}\mathbb{E}_{\mathrm{div}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{M}_{\mathrm{tr}\mathscr{E}}\mathbb{N}_{\mathscr{E}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathrm{tr}\mathscr{S}}\mathbb{N}_{\mathscr{S}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{Bmatrix} \begin{Bmatrix} \vec{\boldsymbol{\omega}} \\ \vec{\boldsymbol{u}} \\ \vec{\boldsymbol{p}} \\ \vec{\boldsymbol{\gamma}} \\ \vec{\boldsymbol{\lambda}} \end{Bmatrix} = \begin{Bmatrix} \mathbb{B}_{\widehat{\boldsymbol{u}}\times\boldsymbol{n}} \\ -\mathbb{M}_{\mathscr{S}}\boldsymbol{f} + \mathbb{B}_{\widehat{p}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

Hybridization

In the mixed formulation, we choose $\mathscr{E}$ to approximate $H_0(\mathrm{curl}, \Omega; \Gamma_\omega)$, $\mathscr{S}$ to approximate $H_0(\mathrm{div}, \Omega; \Gamma_{u\cdot n})$, $\mathscr{V}$ to approximate $L^2(\Omega)$, we obtain

$$\begin{Bmatrix} \mathbb{M}_{\mathscr{E}} & -\mathbb{E}_{\mathrm{curl}}^{\mathsf{T}}\mathbb{M}_{\mathscr{S}} & \mathbf{0} \\ -\mathbb{M}_{\mathscr{S}}\mathbb{E}_{\mathrm{curl}} & \mathbf{0} & \mathbb{E}_{\mathrm{div}}^{\mathsf{T}}\mathbb{M}_{\mathscr{V}} \\ \mathbb{M}_{\mathscr{V}}\mathbb{E}_{\mathrm{div}} & \mathbf{0} & \mathbf{0} \end{Bmatrix} \begin{Bmatrix} \vec{\omega} \\ \vec{u} \\ \vec{p} \end{Bmatrix} = \begin{Bmatrix} \mathbb{B}_{\hat{u}\times n} \\ -\mathbb{M}_{\mathscr{S}}\vec{f} + \mathbb{B}_{\hat{p}} \\ \mathbf{0} \end{Bmatrix}$$

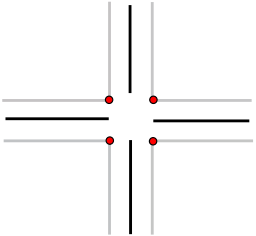
In the hybrid mixed formulation, we further choose $\mathrm{tr}\mathscr{E}$ to approximate $TH_{00}^{1/2}(\Gamma_{mn})$, and $\mathrm{tr}\mathscr{S}$ to approximate $H_{00}^{1/2}(\Gamma_{mn})$. And if we further use dual representations for $\boldsymbol{p}$, γ, $\boldsymbol{f}$ and $\boldsymbol{\lambda}$,

$$\begin{Bmatrix} \mathbb{M}_{\mathscr{E}} & -\mathbb{E}_{\mathrm{curl}}^{\mathsf{T}}\mathbb{M}_{\mathscr{S}} & \mathbf{0} & \mathbb{N}_{\mathscr{E}}^{\mathsf{T}} & \mathbf{0} \\ -\mathbb{M}_{\mathscr{S}}\mathbb{E}_{\mathrm{curl}} & \mathbf{0} & \mathbb{E}_{\mathrm{div}}^{\mathsf{T}} & \mathbf{0} & \mathbb{N}_{\mathscr{S}}^{\mathsf{T}} \\ \mathbb{E}_{\mathrm{div}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbb{N}_{\mathscr{E}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{N}_{\mathscr{S}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{Bmatrix} \begin{Bmatrix} \vec{\omega} \\ \vec{u} \\ \vec{p} \\ \vec{\gamma} \\ \vec{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbb{B}_{\hat{u}\times n} \\ -\vec{f} + \mathbb{B}_{\hat{p}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

Hybridization

$$\begin{Bmatrix} \mathbb{M}_{\mathscr{E}} & -\mathbb{E}_{\text{curl}}^{\mathsf{T}}\mathbb{M}_{\mathscr{S}} & \mathbf{0} & \mathbb{N}_{\mathscr{E}}^{\mathsf{T}} & \mathbf{0} \\ -\mathbb{M}_{\mathscr{S}}\mathbb{E}_{\text{curl}} & \mathbf{0} & \mathbb{E}_{\text{div}}^{\mathsf{T}} & \mathbf{0} & \mathbb{N}_{\mathscr{S}}^{\mathsf{T}} \\ \mathbb{E}_{\text{div}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbb{N}_{\mathscr{E}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{N}_{\mathscr{S}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{Bmatrix} \begin{Bmatrix} \vec{\omega} \\ \vec{u} \\ \vec{\hat{p}} \\ \vec{\hat{\gamma}} \\ \vec{\hat{\lambda}} \end{Bmatrix} = \begin{Bmatrix} \mathbb{B}_{\hat{u}\times n} \\ -\vec{\hat{f}} + \mathbb{B}_{\hat{p}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

Unfortunately, this system is singular, because matrix $\mathbb{N}_{\mathscr{E}}$ is not surjective (breaks the inf-sub condition).

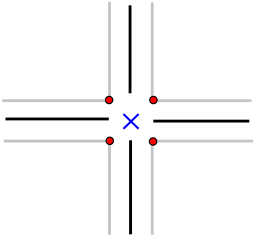


$$\mathbb{N}_{\mathscr{E}} = \begin{Bmatrix} \cdots & \cdots & 0 & \cdots & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & 0 & \cdots & \cdots \end{Bmatrix}$$

Hybridization

$$\begin{Bmatrix} \mathbb{M}_{\mathscr{E}} & -\mathbb{E}_{\text{curl}}^{\mathsf{T}}\mathbb{M}_{\mathscr{S}} & \mathbf{0} & \mathbb{N}_{\mathscr{E}}'^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ -\mathbb{M}_{\mathscr{S}}\mathbb{E}_{\text{curl}} & \mathbf{0} & \mathbb{E}_{\text{div}}^{\mathsf{T}} & \mathbf{0} & \mathbb{N}_{\mathscr{S}}^{\mathsf{T}} & \mathbf{0} \\ \mathbb{E}_{\text{div}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbb{N}_{\mathscr{E}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{D}^{\mathsf{T}} \\ \mathbf{0} & \mathbb{N}_{\mathscr{S}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{D} & \mathbf{0} & \mathbf{0} \end{Bmatrix} \begin{Bmatrix} \vec{\omega} \\ \vec{u} \\ \vec{\hat{p}} \\ \vec{\hat{\gamma}} \\ \vec{\hat{\lambda}} \\ \vec{d} \end{Bmatrix} = \begin{Bmatrix} \mathbb{B}_{\hat{u}\times n} \\ -\vec{\hat{f}} + \mathbb{B}_{\hat{p}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix}$$

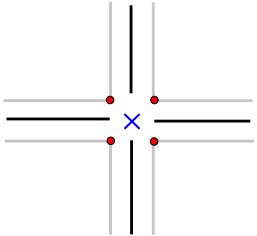
Introduce dummy degrees of freedom :



Hybridization

$$
\left\{\begin{array}{ccc|cc|c}
\mathbb{M}_{\mathscr{E}} & -\mathbb{E}_{\text{curl}}^{\mathsf{T}}\mathbb{M}_{\mathscr{S}} & \mathbf{0} & \mathbb{N}_{\mathscr{E}}'^{,\mathsf{T}} & \mathbf{0} & \mathbf{0} \\
-\mathbb{M}_{\mathscr{S}}\mathbb{E}_{\text{curl}} & \mathbf{0} & \mathbb{E}_{\text{div}}^{\mathsf{T}} & \mathbf{0} & \mathbb{N}_{\mathscr{S}}^{\mathsf{T}} & \mathbf{0} \\
\mathbb{E}_{\text{div}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbb{N}_{\mathscr{E}}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{D}^{\mathsf{T}} \\
\mathbf{0} & \mathbb{N}_{\mathscr{S}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbb{D} & \mathbf{0} & \mathbf{0}
\end{array}\right\}
\left\{\begin{array}{c}
\vec{\omega} \\ \vec{u} \\ \vec{p} \\ \vec{\gamma} \\ \vec{\lambda} \\ \vec{d}
\end{array}\right\}
=
\left\{\begin{array}{c}
\mathbb{B}_{\hat{u}\times n} \\ -\vec{f} + \mathbb{B}_{\hat{p}} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0}
\end{array}\right\}
$$

Introduce dummy degrees of freedom :



$$
\left\{\begin{array}{ccc}
C & D^{\mathsf{T}} & \mathbf{0} \\
D & \mathbf{0} & E^{\mathsf{T}} \\
\mathbf{0} & E & \mathbf{0}
\end{array}\right\}
$$

A condition for such a system being non-singular :

$$\text{Ker}D^{\mathsf{T}} \cap \text{Ker}(E) = 0, \quad \text{Ker}E^{\mathsf{T}} = 0,$$

$$C_{\widetilde{K}\widetilde{K}} : \widetilde{K} \to \widetilde{K} \text{ is non-singular,}$$

$$\text{where } \widetilde{K} := \{\boldsymbol{x}, \text{ such that } \boldsymbol{z}^{\mathsf{T}} D\boldsymbol{x} = 0, \ \forall \boldsymbol{z}^{\mathsf{T}} \in \text{Ker}E\}.$$

Analytical solution

We consider the computational domain $\Omega = [-1, 1]^3$,

$$
\begin{aligned}
u &= \cos(2\pi x)\cos(2\pi y)\sin(2\pi z),\\
v &= \sin(2\pi x)\cos(2\pi y)\cos(2\pi z),\\
w &= -\sin(2\pi x)\cos(2\pi y)\cos(2\pi z) + \sin(2\pi x)\sin(2\pi y)\sin(2\pi z),\\
p &= e^{x(x-1)+y(y-1)+z(z-1)}.
\end{aligned}
$$

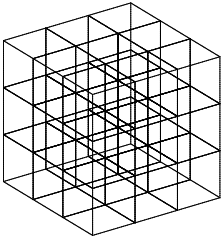


FIGURE – $c = 0$

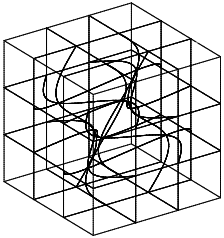
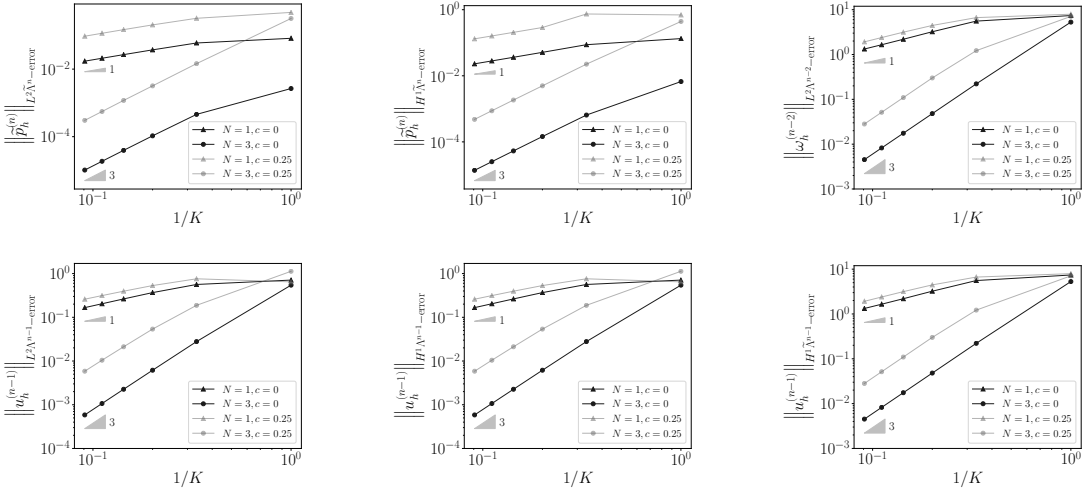
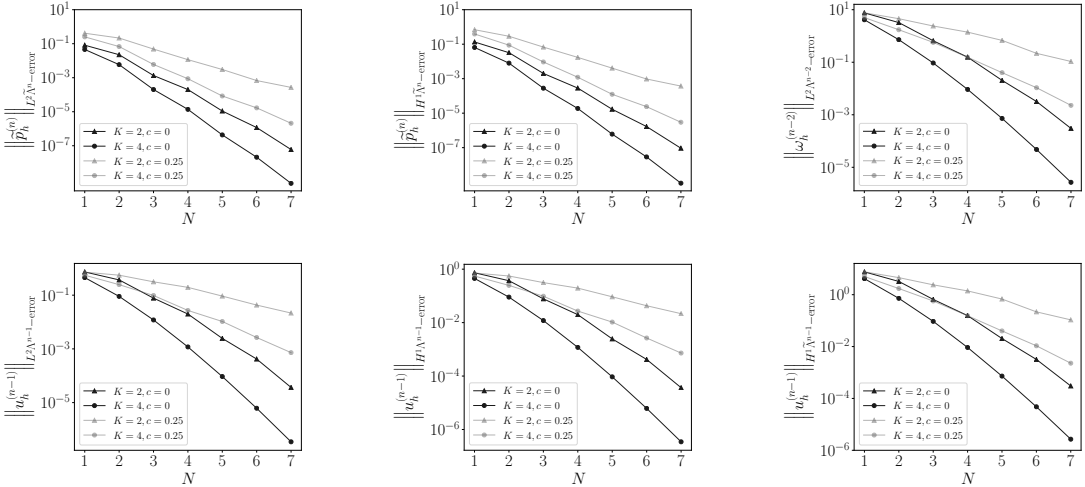


FIGURE – $c = 0.25$

Analytical solution



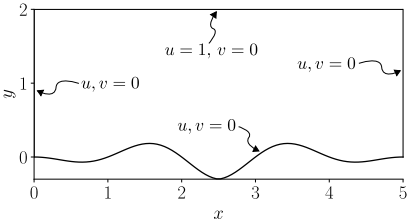
Analytical solution

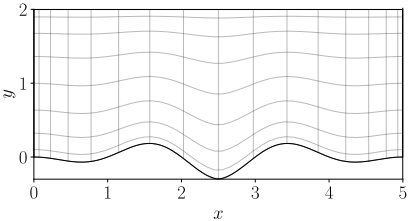


2D wave-shaped lid driven cavity

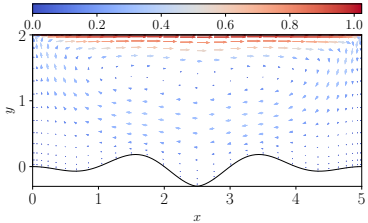
The domain is $(x, y) \in \Omega = [0, 5] \times [y(x), 2]$ where

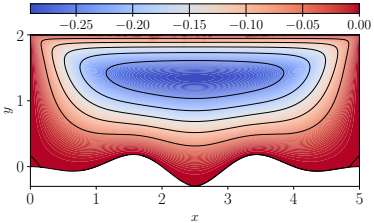
$$y(x) = -0.3 \sin(\pi x) \left(1 - \frac{|x - 2.5|}{2.5}\right).$$

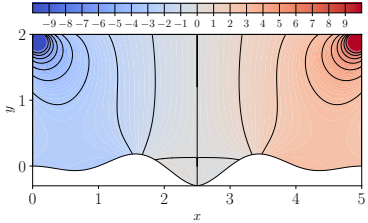


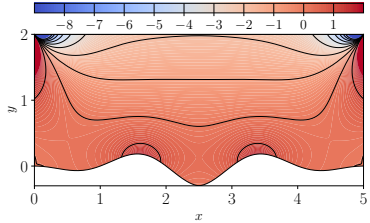


2D wave-shaped lid driven cavity

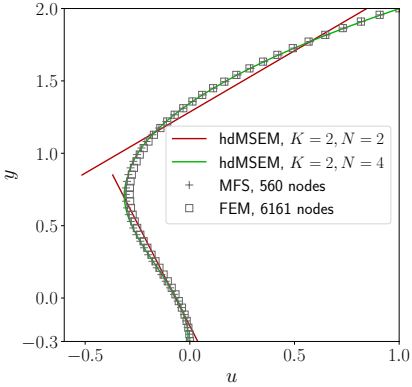




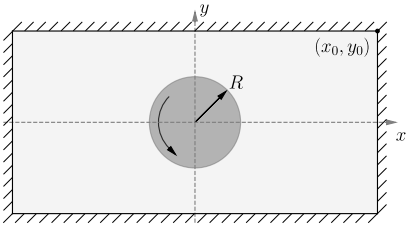




2D wave-shaped lid driven cavity

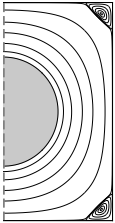


Stokes flow induced by cylinder rotation

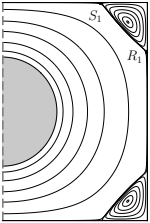


$$A = \frac{x_0}{y_0}$$

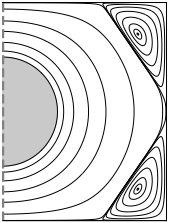
Stokes flow induced by cylinder rotation



$A = 1.00$

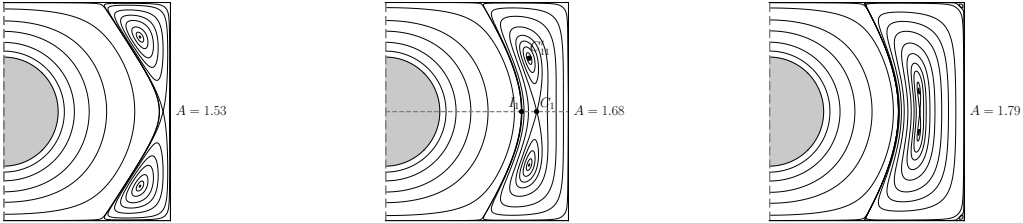


$A = 1.33$

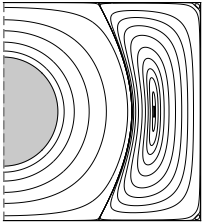


$A = 1.50$

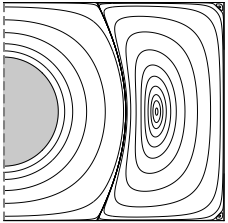
Stokes flow induced by cylinder rotation



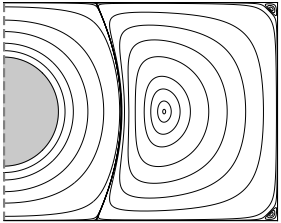
Stokes flow induced by cylinder rotation



$A = 1.81$

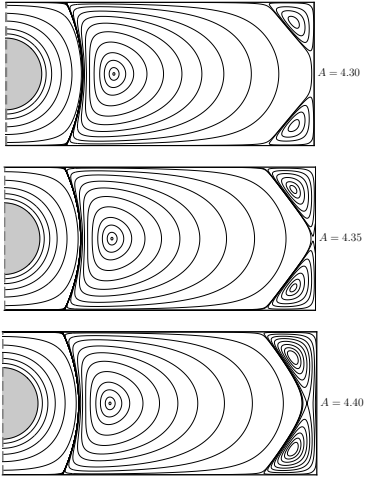


$A = 2.02$

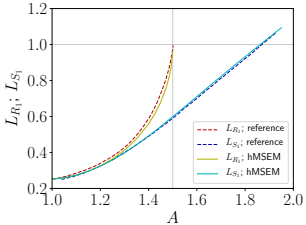


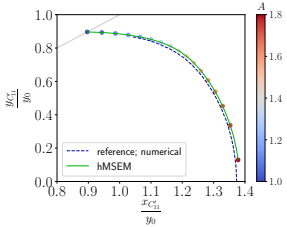
$A = 2.51$

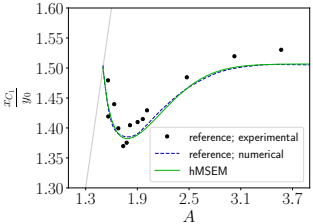
Stokes flow induced by cylinder rotation

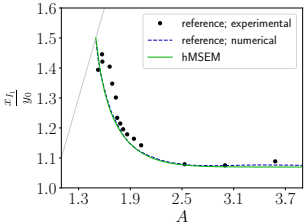


Stokes flow induced by cylinder rotation









Conclusions

We have proposed a high order spectral element method for the Stokes equation in Velocity-Vorticity-Pressure formulation :

- The method uses integral values as dof's.
- The method is hybrid ; it is very easy to parallelize. Imposing boundary conditions is easy ; we have dof's on boundary for both Dirichlet and Neumann boundary conditions.
- The method is mimetic ; first-order differential operators can be preserved at the discrete level.
- The method uses dual polynomials ; some discrete matrices are metric-free, extremely sparse and low order finite-difference(volume)-like (containing non-zero entries of -1 and 1 only).
- It can be efficiently solved by solving a reduced system for the interface variable.

Further developments towards Navier-Stokes is ongoing.

Thanks a lot. Questions ?

Conclusions

We have proposed a high order spectral element method for the Stokes equation in Velocity-Vorticity-Pressure formulation :

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